Chapter 11

The Nature of Space-Time

11.1 The Problem of Coordinates

The basic problem of physics is to track in space and time the development of elements of a system. This requires that we have some method to communicate where and when something took place. In a three dimensional space the place is a set of three numbers; for instance, in a room you could use how far along the floor in a direction along one wall, how far along another wall, and how far up towards the ceiling. The time comes from a clock. This seems so obvious that we generally do not even think about it but, like all the things that we do, this is a subtle operation and we should understand what it is that we are doing when we make a coordinate system. In fact, the realization, that the establishment of the coordinate system is arbitrary is the key to understanding General Relativity. That will come later, Chapter 16 on page 345.

First, lets talk about places. The idea is to label the places. Think of a large parking lot, say at Disney Land. What you need is a unique label for every place. This could be done simply by going around and labeling spots on the lot with the name of a Disney character. This though is not an efficient way to label places. It is a unique label for each place which is how we started but there are many better ways to proceed. For one thing, this labeling scheme does not provide a guide for movement. If you are at Donald Duck, you do not know how far or in what direction to go to get to Goofy, the labels are not an ordered set. You could fix this by ordering the characters alphabetically. This system is nice in that it provides a guide to how to move, it does not indicate how far. It is also not extendable or divisible. An obvious solution is to use as labels the points on the real line, create a mapping of the locations along a direction in the lot with the points of the real line. Since the real line is dense, you can always find a label for any place. If cars suddenly became smaller you would have no problem finding labels. You can also then use these labels to identify direction in the sense that from any location, increasing labels mark one direction along the lot and decreasing mark the other. In other words the sign of the difference between the labels is an indicator of direction of movement. This is a great improvement over the use of Donald Duck to label places.

There are still two problems. First, you need a distance. You can use the length that we discussed in Section 2.3.1 on page 40. In the present case, this means that we define length from how far light travels in a given time or, going back to old fashioned ideas, having some standard rod that can be placed between the points. In the simplest case, you just label the places and then come back later and measure their separation with your standard rod or whatever protocol that is defined for length. In this case the distances between places with the same label difference may have different distances. Don't forget, you just assigned labels from the real line to the places; you just labeled them. This problem though is easy to handle. You just have to measure the separations associated with the different neighboring places. In general, you will not know that all labeled places have the same separations. This process is called establishing a metric on the coordinate system. Our usual use of the cartesian coordinate assumes that when we measure the separations that they are the same in all places, i. e,˙ the underlying manifold is assumed to be homogeneous. The separations are all independent of the labels. Sometimes and in many of the cases that follow this assumption is not warranted.

Secondly, what happens with the idea of extension. What happens when you add to the lot? You have to relabel everything. You can still cover the lot with labels but it is not convenient. By the way, this fact that you can cover a two dimensional space with a wrapped one dimensional label is also a simple proof of the size of the spaces are the same and thus that, although it might appear that a two dimensional infinite space seems bigger than an infinite one dimensional, there are as many points on the plane as there are on a line, see Section **??** on page **??** Thus since you want to extend in a direction that is not along the direction of the chosen sequence, you can improve things quite a bit by having two designators at each place and ordering each of the sets of designators so that a place is a doublet, i. e. (Goofy, Donald Duck).

If you are at the place labeled (3,1) and want to go to the place labeled (7,2) you only need to go four places in the first direction and one place in the other, if you are at (9,0) and want to go to (7,0), you go 2 places in the backwards in the first direction. On the surface of the parking lot though, there are different ways to go between places. An obvious example is to go directly. This is because the two plane is more than two independent lines but accommodates all the paths in between.

In our parking lot, we need two measures of distance, one in each of the independent directions. If both directions are the same, we could generate a combined measure of distance, i.e. not require that all movement be along one of the coordinate directions. More than that if we assume that the space is the same in all directions at any point, isotropic, we can make a measure of distance that is independent of how we chose the directions of the coordinate system. In the case of the parking lot, if we assume that it is isotropic we can adopt for our distance measure $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ where Δx is the displacement in the one direction and Δy is the displacement in the other direction. This distance has the advantage of being independent of the orientation of the axis system. I have to warn you that, if we were really worried about a parking lot, we would most likely not have an isotropic pattern of labels. Automobiles are longer than they are wide.

Again, if at each place the distance algorithm can be the same regardless of where you are the space is homogeneous. If the length scale is also isotropic, you really have the space as described by Descarte and the geometry will be that of Euclid. In general, it could be that, at different places, the distance between places is different or the length is different in different directions. Think about it. On the surface of the earth why should a rod that is held horizontal then turned vertical have the same length? This idea of the distance being the same at all places is also an important simplification, an essential symmetry. When you think about it though it may not be possible. The space may not be homogeneous. Each place may be special. Length may depend on where you are or your orientation. Different directions may have different length scales. In our considerations of General Relativity, Chapter 16 on page 345, these issues will become important.

What is it that we want to get out of this rather extended discussion of the process for labeling a place. The most important thing is the realization that in contrast to what was our original ideas about labeling places, there is a great deal of choice. The choice, as is often the case, is arbitrary and cannot influence important issues. Later, when we discuss the General Theory of Relativity, Chapter 16 on page 345, we will use this ambiguity as a part of the basis for understanding the theory. Suffice it to say, that we must develop a method for labeling places that must be consistent for all observers. It is the consistency requirement that allows us to derive the relationship between the different observers labeling of places and times.

Let us now go into the standard construction of the coordinate system. There are two general methods: the use of confederates at each place and the single observer method. We will start with the confederate method and then show its equivalence to the single observer method which is the one that we will subsequently use.

We begin by defining the spatial coordinates. We assume that we can fill space with a confederate at every place and that the distance between the origin observer and each of the confederates remains fixed. I must warn you about the intrinsic anthropomorphism of this action. Please be assured that the use of words like "confederate" and "observer" which is common to this business imply a humanity that is not really intended. In actually, by confederate or observer, we mean a measuring system – a clock and recording devices – not necessarily a person. It may appear that this assumption about our ability to fill space with fixed and uniform confederates must be true. In fact, one of the insights from general relativity is that this is the case only in the absence of gravity. Since they are fixed in space, we will label the confederate by how far away he/she is in each of the three coordinate directions. Obviously, if the space is homogeneous and isotropic, the location of the origin and the directions of the coordinate axis are arbitrary. For the definition of the distance, we will use the length defined earlier, Section **??** on page **??**, a defined speed of light and a time to label all distances. This speed will be universal for any observer establishing a coordinate system. This means that we need a standard clock and we choose the frequency of given emission line of a Cesium atom. In other words, our second is 9,192,631,770 oscillations of the light. To find the distance to any confederate, we send a light ray to that confederate who reflects it back and, with the standard clock, the observer at the origin can determine how far away that confederate is, $d = \frac{c\Delta t}{2}$, where Δt is the time interval for the round trip of the light.

We have not discussed the problem of labeling the time. The situation is similar to the problem of labeling places. We need some ordered system at each place. What order do a series of events occur in? By endowing each confederate with a clock, we will have at each place a reference set of events to compare with the events being labeled. We use our standard clock. We tell each confederate to make a standard clock. Since the space is assumed to be homogeneous, all the clocks must run at the same rate for each confederate. This is the first step in getting the time of an event that we want to label, to coordinatize. Since we have now endowed each confederate with a clock, we can use as the space and time label for any event as the time recorded on the nearest confederate's clock and the location of the nearest confederate. You should realize that it is not enough to use the same clock at each place but we have to deal with the problem of synchronizing the several clocks; the confederates must synchronize their clocks – at some time agreeing on the time. It must also be consistent with our understanding that the speed of light is the same in all directions regardless of the velocity of the observer. Of course, this leads to the problem of the relativity of simultaneity and makes it important that we understand the process by which any observer synchronizes clocks. For now since we are dealing with only one frame, we do not need to worry about the relativity of simultaneity but it will cause some concern when we compare the coordinate systems constructed by two relatively moving observers. This is discussed in the next section, Section 11.2.2 on page 254. For now, we can accomplish the synchronization by having a burst of light at some very early time released from the origin and, since we know the speed of light and that it is isotropic and we know the location of each confederate, we will know when it passes each confederate and they can set their clocks appropriately.

Let me summarize the confederate scheme for coordinatizing any event, see Figure 11.1 on page 246. An observer establishes a lattice of confederates with identical synchronized clocks and the label of any event in space-time, for that observer, is the reading of the clock and the location of the nearest confederate to that event.

There is a scheme that is equivalent to the confederate scheme that can be accomplished in a less elaborate way by the simple mechanism of having a single clock at the spatial origin and requiring that the observer continuously send out light rays in all directions keeping track of the time of emission. At any event, the incoming light ray is reflected back to the observer. Therefore, the observer has two times and a direction that are associated with any event: the time the reflected ray left and the time of return of the reflected ray and the direction of the reflected light. To yield a spatial coordinatizing that is consistent with the confederate scheme, the spatial distance to the event is the difference in the two times times c divided by 2 or

$$|\vec{x}| = \frac{c(t_2 - t_1)}{2} \tag{11.1}$$

where t_2 is the later time and t_1 is the earlier time. The distance is resolved along the coordinate directions according to the direction of the incoming light ray. To be consistent with the time labeling of the confederate scheme, the time coordinate is

$$t = \frac{t_2 + t_1}{2}. \tag{11.2}$$

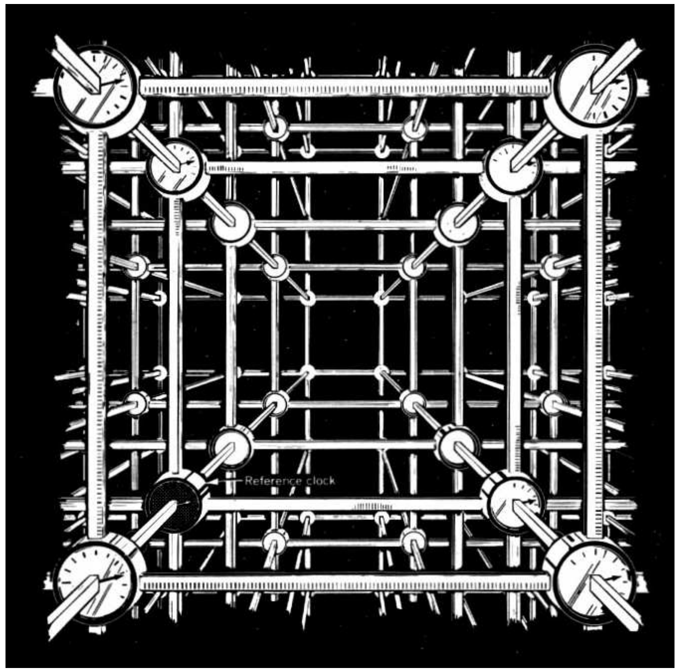

Figure 11.1: **General Construction of a Coordinate System:** Fill all of space with identical clocks. The location of each clock is given and all the clocks are synchronized. An event is given coordinates by assigning the position as the location of the nearest clock and the time on that clock when the event took place.

This protocol for coordinatizing is shown in Figure 11.2 on page 247.

If we accept this protocol for coordinatizing, in order to maintain the equivalence of the inertial observers, all observers should use identical clocks and this protocol. Our problem now becomes the problem of insuring that the clocks are identical and the comparison of results for different inertial observers. These comparative coordinates are related by the Lorentz transformations.

11.2 The Lorentz Transformations

Now that we have developed a protocol for coordinatizing events, we need to find the transformation rules that one inertial observer must use to compare observations with another moving at relative velocity, $\vec{v}$. Actually, this is a special case of the more general problem of finding the transformation rules

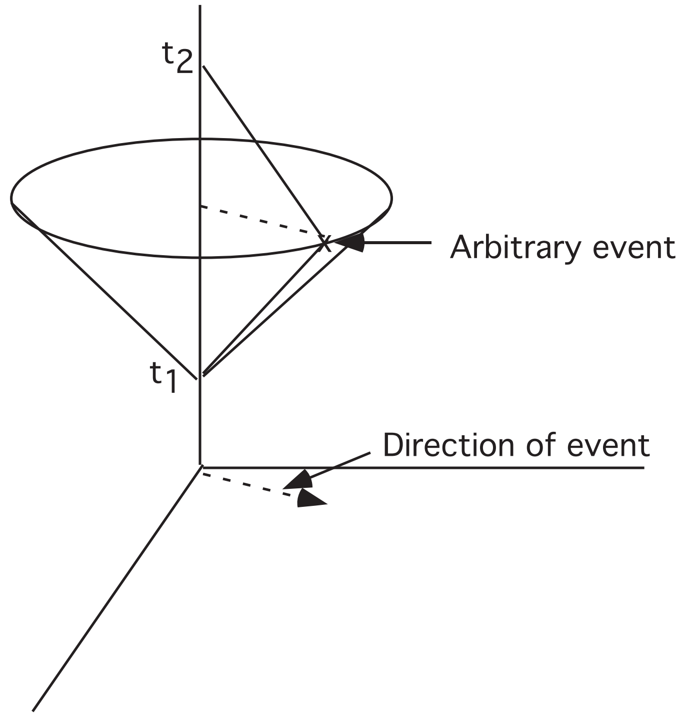

Figure 11.2: **Protocol for Coordinatizing an Event:** The distance of an event for an inertial observer with a clock is $\frac{c(t_2-t_1)}{2}$ and the direction is along the direction of the return signal. The time coordinate is $\frac{t_2+t_1}{2}$.

between any two coordinate systems. Since all inertial observers will see force-free motion as also inertial and as a straight line, you can convince yourself that the most general set of transformation rules between inertial observers is a set of transformations that is linear; it must transform straight lines, one inertial observer, into another straight line, the other inertial observer. Each observer sees his/her time axis as his/her $x = 0$ line.

Before dealing with the case of velocity differences, consider the particularly simple case of two coordinate systems that differ from each other only in the location of the origin. This is the case of two observers that have zero relative velocity, the same orientation of their coordinated axis and it is just the observer one says that observer two has her origin at the location $(x_{2_0}, y_{2_0}, z_{2_0})$. An event measured at the coordinate, $(x, y, z, t)_1$. Will have the label $(x - x_{2_0}, y - y_{2_0}, z - z_{2_0}, t)_2$ to observer two or

$$x' = x - x_{2_0}$$

$$
\begin{aligned}
y' &= y - y_{2_0} \\
z' &= z - z_{2_0} \\
t' &= t
\end{aligned}
\tag{11.3}
$$

This family of transformations has the general name of space translations and is labeled by the values $(x_{2_0}, y_{2_0}, z_{2_0})$. It is an example of a linear transformation between the coordinates; the coordinates enter on both sides of the equations linearly. This example is also inhomogeneous. It has terms that are also independent of the coordinates. The Lorentz transformations that we will deal with here will be linear and homogeneous. Later we will add the inhomogeneous terms which will again deal with translations. The translation transformation were discussed extensively earlier in Section **??** on page **??**. We could also develop the transformation rules for two observers that are at rest with respect to each other, share the same origin, but have different coordinate axis directions. These are the rotation transformations but like the translations these can be incorporated in the family of transformations that are developed here.

Our process for finding the Lorentz transformations will be to use specific rules for the establishment of a coordinate system, see Section 11.1 on page 245, and then to require that the same procedure be used in any inertial system. This process will lead to the fact that for two relatively moving systems, the same event will have two different coordinate designations. This should not come as a surprise since even prior to Einstein's Theory of Special Relativity, the Galilean transformation, see Equation 9.1 on page 216, gave different coordinates for an event when measured by two different inertial observers.

$$
\begin{aligned}
x' &= x - v_{x0} t \\
y' &= y - v_{y0} t \\
z' &= z - v_{z0} t \\
t' &= t,
\end{aligned}
\tag{11.4}
$$

where v_{x0}, v_{y0}, and v_{z0} are the x, y, and z components of the relative velocity of the second observer as measured by the first observer. This family of transformations is labeled by these velocities. As a consequence of these coordinate transformations, the velocities of objects as measured by these observers are also transformed.

$$
v'_x = v_x - v_{x0}
$$

$$
\begin{aligned}
v'_y &= v_y - v_{y0} \\
v'_z &= v_z - v_{z0}
\end{aligned}
\tag{11.5}
$$

These changes also imply that many significant dynamical variables such as momentum and energy are also transformed.

In the case of the Special Theory of Relativity, the rules connecting the different labels for a pair of relatively moving inertial observers that have the same origin and share coordinate axis directions are called the Lorentz transformations. We will derive them in this section. The full family of transformations that include the rotations, translations, and velocity transformations are called the Poincaré transformations.

To construct the Lorentz transformations, we will need to construct two independent inertial coordinate system. It should be clear that each inertial observer must have the same protocol for establishing their coordinate system, the same standard clock, and the same definition of the speed of light. In the previous discussion of Harry and Sally and the the story of the observers in the box car and on the station, Section 10.3 on page 233, we worked for simplicity with only one spatial dimension. Here we will treat the full complication of three space dimensions. Later in many applications, we will return to the case of one spatial dimension where the simplicity allows the point to be made more clearly. You should realize that the primary criteria of the extension to all three spatial dimensions will be that, to each inertial observer, the world should have the usually assumed symmetries of mirror symmetry, inversion symmetry in any direction, and isotropy, no preferred direction.

For definiteness, we will assume that there is an event at which the two relatively moving observers are at the same place and this event will be used as the origin of both coordinate systems. Since as mentioned in the beginning of this section, inertial observers are straight lines in space-time diagrams and thus in three spatial dimensions, only in special cases, will this coincidence occur. Regardless, If this were not the case, a simple spatial coordinate translation of one of the observers, see Equation11.3, on page 248 will relocate the spatial origin.

We set both observer's clocks to $t = 0$ at this event. Since there are two straight lines that meet, there is a plane in space time. The two observers agree that their relative velocity, v, is in that plane and designate the spatial axis in that plane as the positive x axis of observer one. This is the one spatial dimension that will require special attention in the following. Whenever we deal with one space and one time direction, it will be this direction unless stated otherwise. This direction is called the longitudinal direction.

The second observer has chosen the same orientation for his/her x axis.

Firstly, note that the requirement for universal agreement among observers about the speed of light requires that that for both observers light advances by equal distances in equal times. We also use commensurate time and space units. If the spatial intervals are defined the times are in the time that it takes light to travel that distance and visa versa, if the time is the defining unit the distances are the distance that light travels in that time; an example is years and lightyears.

As we saw in Section 10.3 on page 233 the requirement that all observers measure the same speed of light, implies the relativity of simultaneity. Thus although there is agreement about the origin event, $(x = 0, t = 0)$, the locus of events which are straight lines with $t = 0$ to the different observers are different sets of events. This relativity of simultaneity is at the heart of the interpretational difficulties of special relativity.

Since the orientation of the two sets of spatial coordinates is the same, the second observer will say that the first observer has relative speed v directed toward the negative x axis or a velocity of $-v$. This is the direct consequence of the fact that both observers have front back symmetry in this direction and the same speed for light.

First, consider the nature of the agreements and disagreements about measurements that the two observers can have. Both observers are equivalent; neither is preferred. For instance, whatever of substance observer one says about observer two, two must also conclude about one. For instance, if one says that two's standard clock runs the same as one's, then two says that one's clock runs the same. This is the case of Galilean transformations. The two observers would still be equivalent if one said that two's standard clock ran slower if, at the same time, two also said that one's standard clock ran slower. They both disagree in the same way. It would not work that one said that two's clock ran slower and two agreed that his clock ran slower than one's because then they would not be equivalent; one would have the faster clock. An analogy that I like to use is that in the class, all the students are equivalent even in John says that he is sane and the rest of the class is crazy if then Emily is also allowed to conclude that the rest of the class, including John, is crazy and she is sane.

Some coordinates are the same between the two relatively moving observers. Coordinates transverse to the direction of motion are the same. This can be argued this way. Consider two observers as shown in Figure 11.3 on page 251. As stated earlier, the coordinate transformation between these must be linear so that $z' = Bz$, where B is some function of the relative velocity. Now consider the configuration if the two observers had chosen instead a coordinate orientation that is obtained by a rotation about the z axis of π radians and invoking the principal that if one sees two moving along at v along the positive x axis then two sees one as moving along the negative x axis at speed v. This reverses the roles of one and two and thus if the transformation was $z' = Bz$ it is now $z = Bz'$ which implies that $B^2 = 1$. We can dismiss the $B = -1$ solution so that we have $z = z'$. A similar argument can be made for the other transverse direction, the y direction.

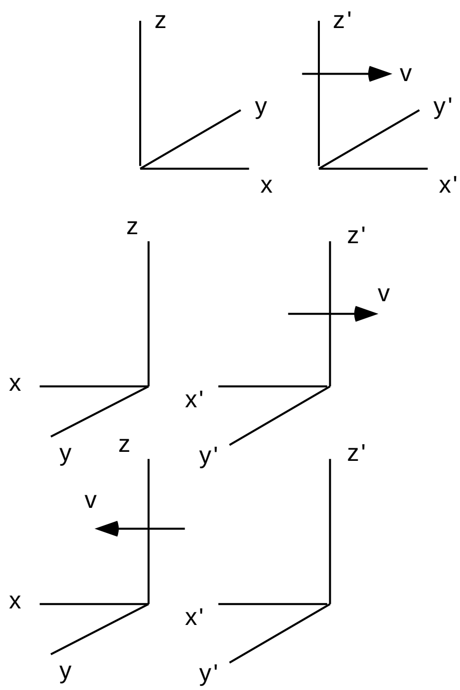

Figure 11.3: **Proof of Agreement on Transverse Direction Coordinates:** At the top of the figure are the coordinate frames for two observers moving relatively along the x axis. Below that are the same observers using frames rotated π radians about the z axis. In the lowest configuration, is the equivalent realization with the first observer moving to the left. This final configuration is the same as the original configuration with the roles of observers one and two reversed.

With the coordinates in the transverse directions the same, we can now show that the relatively moving observers will disagree about the rate at which the standard clock runs.

11.2.1 The Relatively Moving Clock

As discussed in Section **??** on page **??** there is an atomic basis for the standard clock. Regardless, if we can make a system that repeats periodically this system will also be a clock. We will now use the agreement about the transverse lengths to construct a clock that proves that a moving clock must run slower than its identical cousin at rest. Since all observers will agree on the speed of light, we will use the speed of light and an agreed upon distance to make a clock.

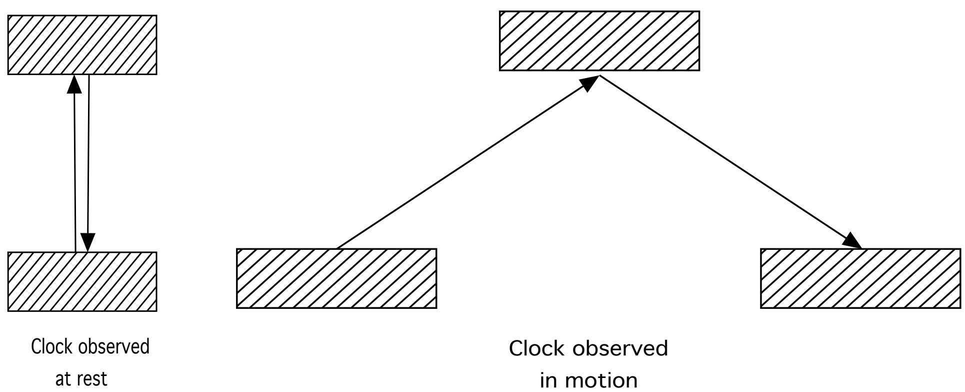

Figure 11.4: **A Clock Using Light:** Using the fact that the speed of light is the same to all inertial observers, we can use light as the basis for a clock. Setting two mirrors a distance, D apart, light bounces back and forth and the interval between passes is the unit of time. Since the light travels a longer distance, this same clock when observed by a relatively moving observer is seen to run slower.

We construct our clock by placing two mirrors a distance D apart and let a burst of light bounce between the two mirrors. The time that passes as the light travels from one mirror to the other and returns is the unit of time. Each observer constructs an identical clock; two mirrors set a distance D apart and held transverse to their relative motion so that they can agree that the mirrors are, in fact, the same distance apart. Consider Harry and Sally again. On her clock, Sally says that the interval between returns of the light is $\Delta t_0 = 2\frac{D}{c}$ but when she observes the operation of Harry's clock, she says that the interval between ticks is longer since the light has to travel a greater distance. Said in another way, only the component of the velocity of the light perpendicular to the mirrors, $v_\perp$ matters. Remember that the speed of light is the same in all directions and that both Sally and Harry have the same speed for light. Thus she says that his clock takes

$$\Delta t = 2\frac{D}{\sqrt{c^2 - v^2}} = 2\frac{D}{c\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (11.6)$$

To Harry though, it is his clock that has a time interval of $\Delta t_0 = 2\frac{D}{c}$ and her clock that is running slow and has the interval $\Delta t = \frac{\frac{2D}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$. Remember that $\sqrt{1 - \frac{v^2}{c^2}}$ is the same for v or $-|v|$.

Look at this situation on space-time diagrams. First we draw the situation as represented by Sally. Here Sally's time axis, her $x = 0$ line is vertical and Harry's time axis is a line with slope $\frac{1}{v}$. If she has a clock that reads at time t_0, she will record a time of $\frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$ for an identical clock carried by him when it reads t_0 to him. She will also record that the moving clock was located at v times that time, $\frac{vt_0}{\sqrt{1-\frac{v^2}{c^2}}}$, since the clock is traveling along Harry's time axis.

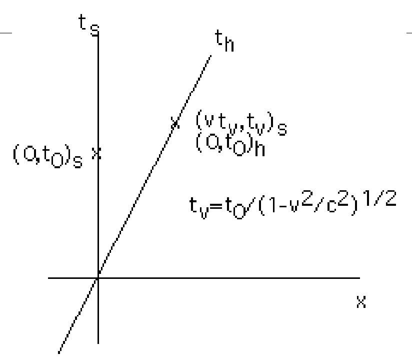

Figure 11.5: **Operation of Mirror Clock:** Sally's time axis is vertical. Harry's time axis has slope $\frac{1}{v}$. If each observer carries an identical clock that to them ticks after a time t_0, the event of the tick on Sally's clock has the coordinates $(0, t_0)_s$ and, since the clocks are identical, the tick of Harry's clock is labeled by Harry as $(0, t_0)_h$. This same event though is labeled by Sally as $(vt_v, t_v)_s$, where $t_v = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

But a similar discussion is appropriate for Harry. He labels the event of that reading on his clock at $(0, t_0)_h$. His coordinates for the event of the reading of t_0 on her clock is at $(\frac{vt_0}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}})_h$. Remember that, to him, Sally's speed is $-|v|$, a negative number, see Figure 11.6 on page 254. The slope of her time axis in his space-time diagram is a negative number, $\frac{1}{v} = \frac{1}{-|v|}$.

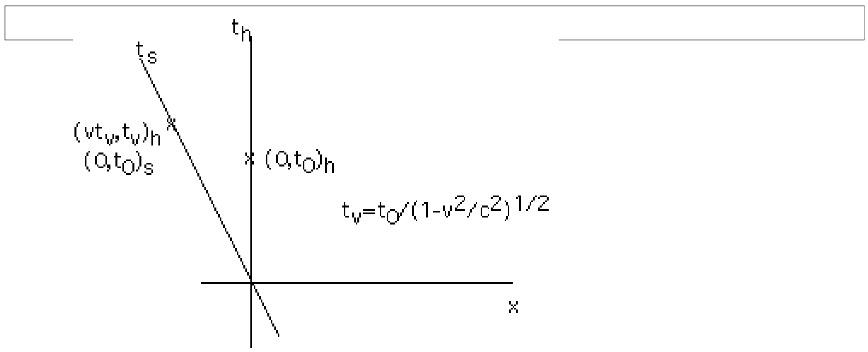

Figure 11.6: **Operation of Mirror Clock in Harry's Frame:** The same pair of related events as in Figure 11.5 on page 253 except as recorded on a space-time diagram based on Harry's time axis being vertical.

11.2.2 Derivation of the Lorentz Transformation

Coordinates of events

As stated in Section 11.1 on page 245, Each observer is to send out a light ray that hits the event and one that returns. Record the times that the first ray is sent out and the time that the second ray comes back and the space coordinate and time coordinate are given by

$$\begin{aligned} x &\equiv \frac{c(\tau_2 - \tau_1)}{2} \\ t &\equiv \frac{\tau_1 + \tau_2}{2}. \end{aligned} \tag{11.7}$$

This rule must be the same for all inertial observers.

When two relatively moving observers label an event, it is important to note though that all observers will use the same two light rays for any particular event, see Figure 11.7 on page 255. In other words, any event is characterized uniquely by the two light rays that pass through it; all observers that are finding the labels of a particular event use the same transmitted and received rays. This apparent coincidence is actually a reflection of the fact that all observers agree on the speed of light and that the intersection of two light rays is an event and thus a unique label of an event.

11.2.3 Details of the Derivation of the Lorentz Transformations

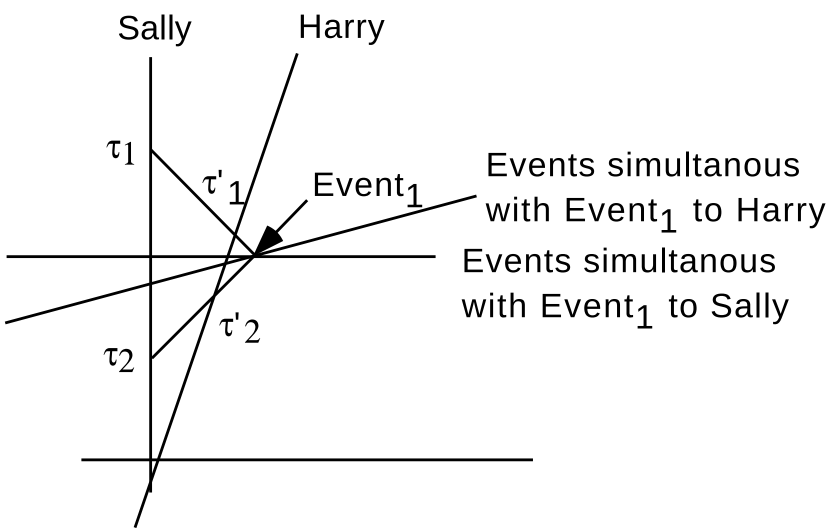

Figure 11.7: **The Rules for Coordinatizing an Event for Two Relatively Moving Observers:** Note that the times t'_1 and t'_2 are the times read on each of the observer's clocks.

Now consider two observers, Sally and Harry, that share the same origin and want to coordinatize the same event. We have shown that the transverse coordinates must be the same for Harry and Sally, Figure 11.3 on page 251, and, in fact, used this information to construct our clocks. Let us now show that this requirement is also obtained in the signaling method of coordinatizing.

In Figure 11.7 on page 255, event 1 is coordinatized by Sally as (x_s, t_s). By definition, Harry would label it (x_h, t_h). The Lorentz transformations are the relationship between (x_s, t_s) and (x_h, t_h).

This is a rather tedious derivation, but a worthwhile exercise. Start by finding the coordinates of the events labeled τ'_1 and τ'_2 in terms of the coordinates of event 1 in Sally's coordinates.

Event τ'_1 has the form (vt_1, t_1) in Sally's coordinates since it is on Harry's time axis and he is moving at a speed v with respect to her. This event is also on a light ray with event 1. The equation of that light ray is

$$x - x_s = c(t - t_s). \tag{11.8}$$

Putting in the coordinates of the event τ_1' which is on this line,

$$vt_1 - x_s = c(t_1 - t_s). \tag{11.9}$$

Solving for t_1,

$$t_1 = \frac{ct_s - x_s}{c - v}. \tag{11.10}$$

Because of time dilation, see Section 11.2.1 on page 252 and Figure 11.5 on page 253,

$$\tau_1' = t_1\sqrt{1 - \frac{v^2}{c^2}}. \tag{11.11}$$

Combining these:

$$\tau_1' = \sqrt{1 - \frac{v^2}{c^2}}\frac{ct_s - x_s}{c - v} \tag{11.12}$$

Similarly for event τ_2'

$$\tau_2' = t_2\sqrt{1 - \frac{v^2}{c^2}} \tag{11.13}$$

and

$$\tau_2' = \sqrt{1 - \frac{v^2}{c^2}}\frac{ct_s + x_s}{c + v} \tag{11.14}$$

Inserting this into the definitions, Figure 11.7 on page 255 and Equation 11.7 on page 254, and doing some straightforward algebra, we have

$$\begin{aligned} x_h &= \frac{x_s - vt_s}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t_h &= \frac{t_s - \frac{v}{c^2}x_s}{\sqrt{1 - \frac{v^2}{c^2}}}, \end{aligned} \tag{11.15}$$

which are the appropriate Lorentz transformations for this case.

Adding the fact that the transverse directions are unaffected by the velocity transformation, we get the usual Lorentz transformations, Equation 10.8

on page 239, or written out more fully,

$$\begin{aligned} x_h &= \frac{x_s - vt_s}{\sqrt{1-\frac{v^2}{c^2}}} \\ y_h &= y_s \\ z_h &= z_s \\ t_h &= \frac{t_s - \frac{v}{c^2}x_s}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned} \tag{11.16}$$

An interesting feature of these relations is that the combination $(ct_h)^2 - (x_h)^2 - (y_h)^2 - (z_h)^2$ does not involve the velocity and is therefore equal to Sally's coordinates for the same event,

$$(ct_h)^2 - (x_h)^2 - (y_h)^2 - (z_h)^2 = (ct_s)^2 - (x_s)^2 - (y_s)^2 - (z_s)^2. \tag{11.17}$$

This is a special case of the general form for the invariants of the Lorentz transformations, see Section 12.3 on page 280. We will take advantage of this simple relationship in our subsequent analysis of these relationships.

It is worthwhile checking to see if the Lorentz transforms effect lines of simultaneity and observer time axis as expected. For instance, Harry's line of simultaneity with the origin is the set of events at $t_h = 0$ which is also the events $t_s - \frac{v}{c^2}x_s = 0$; a line slopped at $\frac{v}{c^2}$ through the origin. Harry's time axis is his $x_h = 0$ line. This is a line through the origin with slope $\frac{1}{v}$ on Sally's space-time diagram.

11.3 Using Lorentz Transformations

11.3.1 Time Dilation

Time dilation is the general term for difference in the time interval recorded on two relatively moving but otherwise identical clocks. We had already treated the problem of time dilation in Section 11.2.1 on page 252 using a light clock with mirrors but this is a general phenomena and not limited to light clocks and, using invariants of Equation 11.17 on page 257, the formula for the time difference is direct and intuitive.

If you are moving with a clock and it reads an interval of time Δt_0. Say this is time of a tick on Sally's clock. At the instant of the tick of Sally's clock, an identical clock which is moving uniformly at a speed v relative to her and synchronized with her clock at the start of the interval by Harry, will read a time Δt which is less than Δt_0, see Figure 11.8 on page 258 This

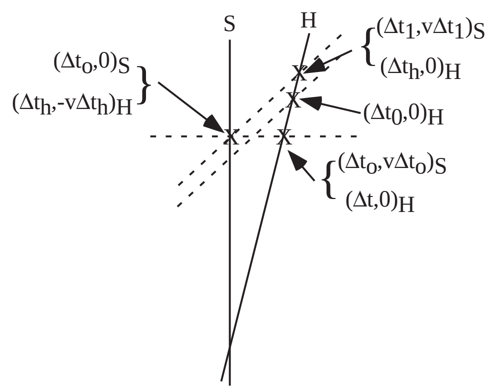

Figure 11.8: **Time dilation in a Moving Clock** Two observers, Sally and Harry, with identical clocks are moving relative to each other at a speed v. At the time that the one observer, say Sally, notes the time Δt_0 on her clock she would assign the coordinates of the simultaneous event on the other clock as $(\Delta t_0, v\Delta t_0)_S$. The observer moving with that clock, Harry, records the event as $(\Delta t, 0)_H$. Since the invariant form must take on the same value for all Lorentz equivalent coordinatizings of the same event, $(\Delta t)^2 - (\frac{0}{c^2})^2 = (\Delta t_0)^2 - (\frac{v\Delta t_0}{c^2})^2$ or $\Delta t = \sqrt{1 - \frac{v^2}{c^2}}\Delta t_0$. Similarly, setting Harry's time for the tick of Sally's clock as Δt_h. his coordinates for the tick of Sally's clock is $(\Delta t_h, -v\Delta t_h)_H$. The invariant relationship requires $\Delta t_h = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

same event has two Lorentz equivalent coordinate descriptions, $(\Delta t_0, v\Delta t_0)_S$ and $(\Delta t, 0)_H$. Therefore the invariant requires that

$$\Delta t = \sqrt{1 - \frac{v^2}{c^2}}\Delta t_0. \tag{11.18}$$

Thus since $\Delta t < \Delta t_0$, Sally says that Harry's clock runs slower. To Harry his clock ticks after a time Δt_0 and thus occurs on his time axis after this event which Sally says is simultaneous with her clock tick.

The inverse problem of when Harry says that Sally's clock has ticked requires that we find the event on Harry's time axis simultaneous with the tick of Sally's clock. Harry, assigns a coordinate time of Δt_h to this event, see Figure 11.8 on page 258. Since Sally is moving with a speed v in the negative position direction, Harry assigns the coordinate designation of $(\Delta t_h, -v\Delta t_h)$ to the event of Sally's clock ticking at her clock. Again the invariant requires that $\Delta t_0 = \sqrt{1-\frac{v^2}{c^2}}\Delta t_h$ and, in this case, $\Delta t_h > \Delta t_0$. Since

$$\Delta t_h = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} > \Delta t_0 = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} > \Delta t,$$

Harry says that Sally's clock has not yet ticked when his clock reads Δt_0; it runs slower and yet Sally says that her clock if the first to tick.

Although at first this seems to be an anomaly, with some thought it is clear that this is the way it has to be. Either all identical clocks indicate the same time intervals which is the Newtonian case or as, in this case, all relatively moving clocks run slow but each clock unto itself is correct. It is like a world in which I am sane and everyone else is crazy. This is an equivalent relationship is it holds for everyone. Thus others would conclude that, although I think otherwise, they are sane and I am among the crazies. Of course, a situation with moving clocks running faster would be equivalent but this is not what nature choses.

11.3.2 Length contraction

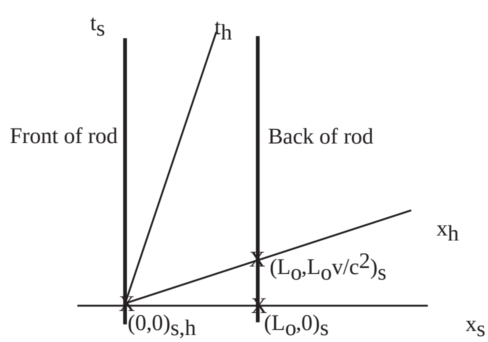

Figure 11.9: **Length Contraction:** Sally carries a rod of length L_0. The ends of the rod are indicated by the vertical lines $x_s = 0$ and $x_s = L_0$. Harry's time axis is labeled t_h. Each observer says that the length of the rod is the separation of the ends at the same time. Due to the relativity of simultaneity, they use different event pairs to measure a length. It should therefore not come as a surprise that they get different lengths.

In the transverse direction there is no ambiguity about length. In the direction of the motion we have to be careful. Sally holds a rod of length

L_0. To Harry who is moving relative to Sally at speed v along the same direction as the extended rod, how long is the rod?

In order to understand the situation, let's look at a space-time diagram, Figure 11.9 on page 259.

To any observer, the length of a rod is where the ends are at the same time to that observer. In the frame that is commoving with the rod, Sally, the ends of the rod are a the two lines $x_s = 0$ and $x_s = L_0$. Thus two events at the ends of the rod that are simultaneous to Sally are $(0,0)_{s,h}$ and $(L_0,0)_s$ and thus the length is the difference in the space coordinates or L_0. To Harry, the event that is simultaneous with $(0,0)_{s,h}$ is and on the other end of the rod and is $(L_0, \frac{L_0 v}{c^2})_s$. Remember that Harry's line of simultaneity has slope $\frac{v}{c^2}$. Using the Lorentz transformations to get Harry's coordinate assignment for this event, $(L_0, \frac{L_0 v}{c^2})_s$ is transformed to $(L',0)_h$, where $L' = \sqrt{1-\frac{v^2}{c^2}}L_0$. Thus to Harry the length of the rod is the difference of the space coordinates at the same time or he says that the length is

$$L' = L_0\sqrt{1-\frac{v^2}{c^2}} \tag{11.19}$$

Another way to see the same result is to realize that to Harry, the coordinates of the end of the rod at $t_h = 0$must take the form $(L',0)$. Sally's coordinates for that same event are $\left(L_0, L_0\frac{v}{c^2}\right)$. Harry's coordinates for any event differ from Sally's by a Lorentz transformation. Using the invariant of the Lorentz transformations, $(0)^2 - (L')^2 = (c\frac{L_0 v}{c^2})^2 - (L_0)^2$ which then gives Equation 11.19 on page 260 for L'.

11.3.3 The Doppler Effect

We are all familiar with the classical Doppler effect. An approaching fire truck is racing to the chemistry building and the siren is at a high pitch. When the fire truck passes and is moving away from us, the pitch of the siren drops. In other words, an approaching sound source souds at a higher frequency than the frequency that it produces and a receding sound source has a lower frequency than that of the source.

Consider the case of Sally moving by Harry at a speed v. Sally sends a ray of light to Harry at a time τ_e after the time of their coincidence. The event of arrival of the light on Harry's space-time diagram of the emission is at some time t_e at location $x_e = vt_e$ since it is on Sally's time axis which goes through the origin event and has slope $\frac{1}{v}$. Since she would coodinatize the event as

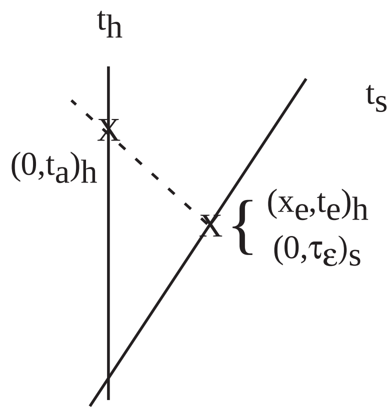

Figure 11.10: **Doppler Effect:** After passing a light signal is sent between two relatively moving observers, Harry and Sally with Sally transmitting. The time interval, t_a, between their passing and the arrival of light signal from the other observer who transmitted at a time τ_e after passing is $t_a = \sqrt{\frac{(1+\frac{v}{c})}{(1-\frac{v}{c})}}\tau_e$

$(0, \tau_e)$, we can use the invariant $c^2\tau_e^2 = c^2t_e^2 - v^2t_e^2$ to find the relationship between τ_e and t_e or as expected from time dilation $\tau_e = t_e\sqrt{1-\frac{v^2}{c^2}}$.

We can find the time of arrival of the light ray emitted by Sally to Harry, t_a,. Note that from Figure 11.10 on page 261, Sally is moving away from Harry. We use the equation of the light ray going through the emission event. The equation of this line is $(x - x_e) = -c(t - t_e)$. Thus $ct_a = ct_e + x_e = (c+v)t_e$ or $t_a = \frac{(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}}\tau_e = \sqrt{\frac{(1+\frac{v}{c})}{(1-\frac{v}{c})}}\tau_e$. τ_e could be considered the first of a sequence of periodic signals and t_a the interval of between the reception of a pair of the signals. Thus the frequency of emission, $f_e = \frac{1}{\tau_e}$ and the frequency of reception, $f_a = \frac{1}{t_a}$ are related by

$$f_a = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}f_e, \tag{11.20}$$

which is the relativistic Doppler effect for the frequency of a signal sent from a receding transmitter to a receiver. The non-relativistic limit, $\frac{v}{c} \ll 1$, of this expression yields the usual Doppler formula, $f_a = \left(1 - \frac{v}{c}\right)f_e$. The case of the approaching emitter is simply found by replacing v by $-v$.

There may be some concern about the fact that, in a situation in which there is more than one spatial dimension, two inertial observers may not meet and this derivation used their coincidence event as a basis. Remember that in any number of spatial dimensions, there is always an event pair that are events of closest approach between the observers. If a commover to one of the observers, O_1, is located at the event of closest approach on the other observer, O_2, the above analysis works for that commover. That commover sees the frequency given by Equation 11.20 on page 261. That comover can then merely retransmits the received signals to O_1. Of course, there is no difference in the time interval for signal between the commover and O_1. Thus O_1 will see that the interval given by Equation 11.20 on page 261. When you think about this problem you realize that the resolution is in the translation symmetry of the individual inertial observers.

11.3.4 Addition of velocities

Given the Lorentz transformations, Equation 11.16 on page 257, it is now easy to get the formula for the addition of velocities. Consider Harry, Sally and Tom. Harry moves by Sally to increasing x at v_{h_s}, where v_{h_s} is Harry's velocity as measured by Sally. For simplicity of the analysis, first let's consider that case that from Sally's point of view Tom is also moving in the positive x direction and he moves by Sally at v_{t_s}, where v_{t_s} is Tom's velocity as measured by Sally. How fast does Harry say that Tom is moving? The situation is shown Graphically in Figure 11.11 on page 262.

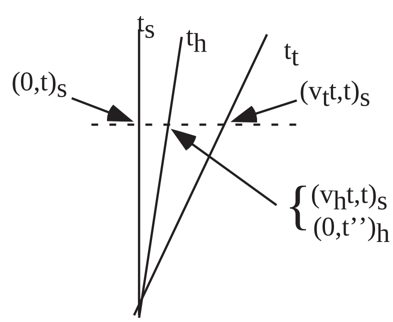

Figure 11.11: **Addition of Velocities.** To determine how colinear velocities add, consider three inertial observers, Harry, Sally, and Tom moving in the same direction. If we know Harry's and Tom's velocities relative to Sally, we can find Tom's velocity relative to Harry by transforming to Harry's frame.

Although we do not need it, note that $t'' = \sqrt{t^2 - \frac{v_{h_s}^2 t^2}{c^2}} = \sqrt{1 - \frac{v_{h_s}^2}{c^2}}\ t$.

Also note that the $v_t$and $v_h$in the figure should be v_{t_s} and v_{h_s}. The graphics package does not allow for stacked subscripts.

Drawing this same set of events in terms of a coordinate system based on Harry is given in Figure 11.12 on page 263

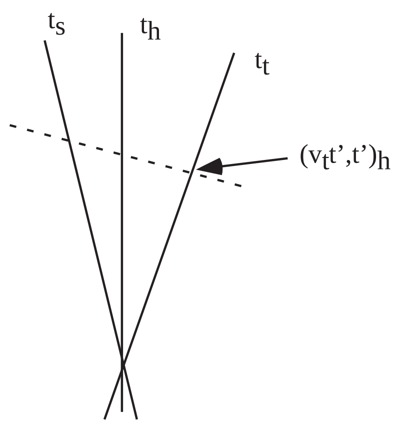

Figure 11.12: **Addition of Velocities from Harry's Point of View** Three inertial observers moving colinearly as described in Figure 11.11 on page 262 in a coordinate system using Harry as the vertical time axis.

Similarly to above the v_t in the figure should be v_{t_h}. Using the Lorentz transform for this event in Harry's coordinates.

$$v_{t_h}t' = \frac{v_{t_s}t - v_{h_s}t}{\sqrt{1-\frac{v_{h_s}^2}{c^2}}}$$
$$t' = \frac{t - \frac{v_{h_s}}{c^2}v_{t_s}t}{\sqrt{1-\frac{v_{h_s}^2}{c^2}}}$$

Dividing these equations

$$v_{t_h} = \frac{v_{t_s} - v_{h_s}}{1 - \frac{v_{h_s}v_{t_s}}{c^2}} \quad (11.21)$$

In the limit of $\frac{v}{c}$ small, this result reduces to the usual Galilean result, Equation 11.5 on page 249.

There should be some concern about how general this result is. Even in a situation with one spatial dimension, there is no need for an event of coincidence between the three observers. There could be situations with a coincidence of Harry and Sally and a different event for the coincidence of Harry and Tom and of Sally and Tom. The above proof will still work for a comover of Sally at the event of coincidence of Harry and Tom and, since v_{h_s} and v_{t_s} is the same for this comover, the result for the relative velocity of Tom to Harry, v_{t_h}, the desired result, is still that given by Equation 11.21 on page 263. Another approach to the resolution of this problem is to realize that we enjoy space and time translation symmetry. Even if the events of coincidence of Harry and Sally and Harry and Tom and Tom and Sally are not the same event, we and translate such that all three events of two party coincidence are the same. The alert reader should understand that these two explanations of the velocity addition argument are actually the same.

A more substantive concern is that in higher dimensions there is no need for any coincidences at all but also that the velocities need not be co-linear. To be specific, at some time t_{0_s} to Sally, Harry and Tom are at some distance, $\vec{x}_{h_s}$ and $\vec{x}_{t_s}$, with velocities $\vec{v}_{h_s}$ and $\vec{v}_{t_s}$ relative to Sally. There exist comovers of Harry and Tom at Sally at this time. We can now call this the event of coincidence and find the relative velocity of these co-movers. The relative velocity between these co-movers will be the same as the relative velocity for Tom relative to Harry. To find this relative velocity, Sally can now do a similar exercise to the one above: pick a time, t, and label where Harry's co-mover and Tom's co-mover are, Lorentz transform to the frame that moves Harry's comover to origin for all times and then find Tom's co-movers relative velocity from his coordinates. The difference between this case and the above is that, after identifying the co-movers at the coincidence point, the co-movers velocities relative to Sally, $\vec{v}_{h_s}$ and $\vec{v}_{t_s}$, are not necessarily co-linear. Using the isotropy of Sally, we can orient the x axis along the velocity of Harry's co-mover. Similarly, we can orient the frame so that the Tom's co-mover velocity is the in the $x-y$ plane. Thus the general case is reduced to one requiring only two spatial dimensions and an analysis which is similar to the one in Figure 11.11 on page 262 and Figure 11.12 on page 263 but now in two spatial dimensions. For generality, let's call the x direction the longitudinal direction and y the transverse direction. This construction is shown in Figure 11.13 on page 265.

Again, in the figure $(v_{Lt}t', v_{Tt}t', t')_h$ should be $(v_{Lt_h}t', v_{Tt_h}t', t')_h$ but due to limitations of the graphics package could not be double subscripted.

Using the appropriate Lorentz transformation,

$$v_{Lt_h}t' = \frac{v_{Lt_s}t - v_{h_s}t}{\sqrt{1-\frac{v_{h_s}^2}{c^2}}}$$

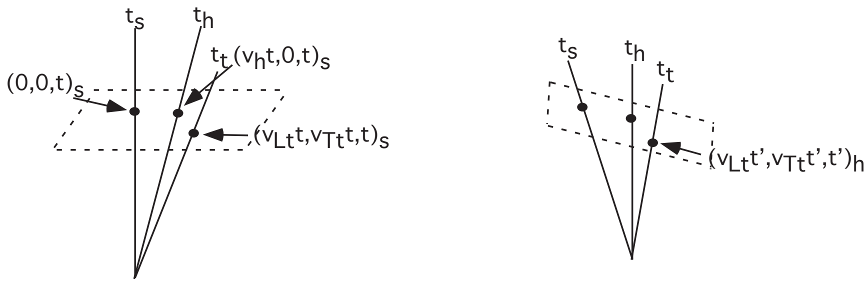

Figure 11.13: **Addition of Velocities for Non-Co-linear Case:** Three inertial observers moving non-co-linearly in two coordinate systems, one using Sally as the vertical axis and one using Harry as the vertical time axis.

$$\begin{aligned} v_{Tt_h} t' &= v_{Tt_s} t \\ t' &= \frac{t - \frac{v_{h_s}}{c^2} v_{Lt_s} t}{\sqrt{1 - \frac{v_{h_s}^2}{c^2}}} \end{aligned} \tag{11.22}$$

Thus, we see that the longitudinal component transforms as in the one space dimension case, Equation 11.21 on page 263, or

$$v_{Lt_h} = \frac{v_{Lt_s} - v_{h_s}}{1 - \frac{v_{h_s} v_{Lt_s}}{c^2}}. \tag{11.23}$$

The transverse component also changes and is given by

$$v_{Tt_h} = \frac{v_{Tt_s} \sqrt{1 - \frac{v_{h_s}^2}{c^2}}}{1 - \frac{v_{h_s} v_{Lt_s}}{c^2}}. \tag{11.24}$$

Despite the added complications, in the limit of $\frac{v}{c} \rightarrow \infty$, this result reduces to the usual Galilean result, Equation 11.5 on page 249.

11.3.5 Time for Different Travelers

Sally and Dorothy are inertial and are at rest with respect to each other, co-moving. They are separated by a distance of one light year. Harry is traveling at $\frac{3}{5}$c toward Sally and Dorothy. He passes Sally and continues to Dorothy, turns around instantly and at the same speed goes back to Sally. How long is the trip from Sally and back to Sally according to Sally? According to Harry? How far apart are Sally and Dorothy according to Harry? When he is at Sally, how far away does he say that Dorothy is?

Sally says that Harry reached Dorothy in $\frac{5}{3}$ years. Dorothy was one light year away and he was traveling between them at $\frac{3}{5}$c. Similarly for the return trip. So she says that the round trip takes $\frac{10}{3}$ years. By Sally's coordinatizing, the event of Harry meeting Dorothy is $(1, \frac{5}{3})_s$. Using the Lorentz transformations, this same event is labeled by Harry as $(0, \frac{4}{3})_h$. By the way, although Sally says that she and Dorothy are one lightyear apart, he says that they are $\frac{4}{5}$ of a lightyear apart, see Section 11.3.2 on page 259. (He says that Dorothy is coming at him at $\frac{3}{5}$c and it takes $\frac{4}{3}$ of a year for her to get there.) On a space time diagram, the event of his being at Sally for the first time and the event of where Dorothy is are different for Harry and Sally because of the relativity of simultaneity, Section 10.3 on page 233. To Harry the return trip to Sally will also take $\frac{4}{3}$ of a year and thus the round trip time is $\frac{8}{3}$ of a year. In other words, Harry and Sally disagree about the elapsed time of the trip.

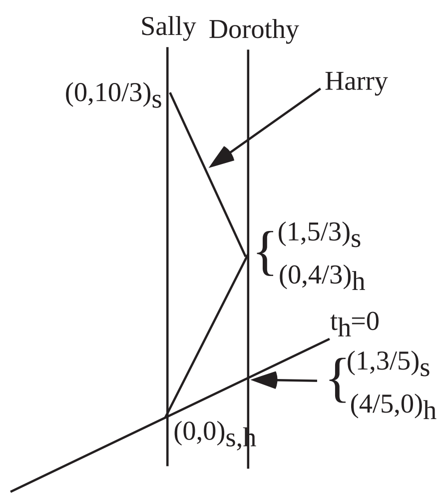

Figure 11.14: **Harry and Sally over Different Trajectories:** Harry travels between Sally and Dorothy. Sally and Dorothy are at rest relative to each other and separated by one light year. Harry departs from Sally and returns after meeting Dorothy. He travels at $\frac{3}{5}$c to and from Sally. If Harry and Sally measure the elapsed time, they disagree about the total time of the trip.

This difference between elapsed times on different trajectories is a general feature of Special Relativity. Before we can discuss this issue in general terms, we will need to develop an appropriate vocabulary.

11.3.6 Visual Appearence of Rapidly Moving Objects

In order to find the apparent length or the length as it is seen, we must realize that seeing involves the light that enters the eye at any instant. Thus the events of interest are those that leave the extended body at different times and are thus on light-like intervals, on the light cone from the observation event. The following figure, again showing only the ends of the rod, for our case of the relative velocity of $\frac{3}{5}c$, indicates the event at the far end of the rod that is seen at the same time as the origin event at the near end.

The space-time diagram is shown in Figure 11.15 on page 267. This diagram makes clear what is meant by the apparent length of the longitudinally moving rod. Of course, for longitudinal orientation, the rod is always seen as a point. Its apparent length is the range of coordinates covered by the rod at the time the near end is observed. We can find this length directly using

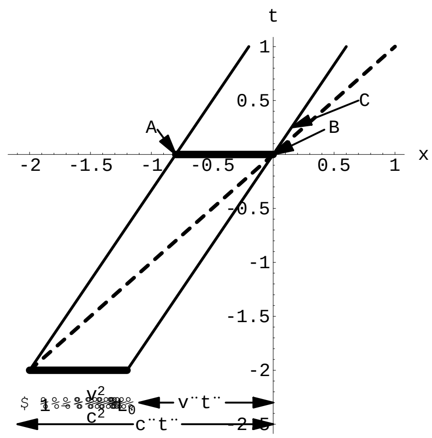

Figure 11.15: **Space-time diagram of moving rod** Space-time diagram in frame of original observer showing the ends of the rod moving at $\frac{3}{5}c$ and the events on the light ray, shown dotted, that are on the same ray as the origin event. The two horizontal lines at $t = 0$ and $t = -2$ show the position of the rod at those times to the rest observer. Shown below that is the length of the rod, $\sqrt{1 - \frac{v^2}{c^2}}$, the distance to the front of the rod, $v|t|$, and the distance to the back of the rod, $c|t|$, to the rest observer.

the space-time diagram. Doing the general case, the equation of the light ray linking with the origin event, B, is $t = \frac{x}{c}$. The time axis of the far side of the rod is $t = \frac{1}{v}(x + \sqrt{1 - \frac{v^2}{c^2}} L_0)$. Finding the simultaneous solution to these two equations, this event is thus $(-\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}L_0, -\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\frac{L_0}{c})$. The definition of the apparent or visual length is the spatial coordinate difference between these two light like related events, B and this event, or $L_{vis} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}L_0$. Thus the longitudinally moving rod actually can appear longer than the rod at rest. A similar analysis for the rod once it has passed the observer yields for the visual length $L_{vis} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}L_0$. In this case, the length is contracted.

This is a rather striking result. First, since it is first order in $\frac{v}{c}$, it is a large effect. The Lorentz-Fitzgerald contraction is second order. Also since it is first order, it is sensitive to the sign of v. Thus rods moving toward the observer are stretched and rods moving away are contracted. This is consistent with our understanding of the basis for the effect. Because of the finite speed of light, we see farther parts of an extended object at times that are earlier than the times that we see the near parts. Thus, for the rod moving toward you, the farther part is seen when it is further away. Whereas, for the receding rod, the farther parts are not as far away. With this observation and from the form of the equation, we realize that this effect is the spatial correspondent to the well-known Doppler effect for temporal differences. In that case, the approaching light intervals are shortened and the frequencies go up and for a receding source the intervals get longer and the frequencies go down. Here the expansion and contraction are reversed. There should be no surprise that there is a spatial correspondent to the Doppler effect.

The case of the transverse rod can be analysed in a similar fashion.

The visual appearance of a rapidly moving object was discussed first by Penrose [**?**] and elaborated by Terrell [**?**]. They discuss the case of an object that is not moving longitudinally toward the observer and restrict the analysis to objects with a small viewing aperture. A very clear presentation of their arguments is give by Weisskopf [**?**]. The case of the longitudinally oriented rod is discussed by Weinstock [**?**] but not using space-time diagrams.